# Midterm Solutions

The Limits of Logic

Due 2pm Oct 18, 2023

**Turn in answers to any five out of the following six problems.** Each problem is worth 8 points, for 40 total points possible.

Explain your reasoning clearly for each question. You can use any of the facts we've already proved in class without further justification, but please be explicit about which facts you are using.

## Problem 1

Let $A$ be a set, and let $X$ be a set of subsets of $A$. (That is, $X \subseteq PA$.) We define the relation $\sqsubseteq$ as follows:

> For any $a, b \in A$, let $a \sqsubseteq b$ iff, for every set $U \in X$, if $a \in U$, then $b \in U$.

(a) Prove that for any $a, b, c \in A$, if $a \sqsubseteq b$ and $b \sqsubseteq c$ then $a \sqsubseteq c$.

(b) Suppose that for any $a, b \in A$, there is some $c \in A$ such that we have both $a \sqsubseteq c$ and $b \sqsubseteq c$. (This is called the *directed set property.*) Prove that any two non-empty sets $U, V \in X$ have some element in common: in other words, the intersection $U \cap V \neq \varnothing$.

## Solution

(a) Let $a, b, c \in A$, and suppose $a \sqsubseteq b$ and $b \sqsubseteq c$. Let $U \in X$ and suppose $a \in U$. Then since $a \sqsubseteq b$, by the definition of $\sqsubseteq$ we must have $b \in U$. Similarly, since $b \sqsubseteq c$, by definition $c \in U$. So for any $U \in X$ such that $a \in U$, we also have $c \in U$. That is, $a \sqsubseteq c$.

(b) Let $U, V \in X$ be non-empty sets: that is, each has at least one element. Let $a \in U$, and let $b \in V$. Then by the directed set property, there is some $c \in A$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$. By the definition of $\sqsubseteq$, since $a \in U \in X$ we must have $c \in U$, and since $b \in V \in X$ we must have $c \in V$. So the intersection of $U$ and $V$ is non-empty.

Problem 2

(a) Write down a recursive definition of a function echo : $\mathbb{S} \to \mathbb{S}$ that takes each string to a corresponding string with each symbol repeated. For example:

$$\begin{aligned} \text{echo}\,\texttt{ABC} &= \texttt{AABBCC} \\ \text{echo}\,\texttt{AAB} &= \texttt{AAAABB} \\ \text{echo}\,\texttt{BAAAC} &= \texttt{BBAAAAAACC} \\ \text{echo}() &= () \end{aligned}$$

(b) Use your definition to prove by the following by induction:

$$\text{echo}(s \oplus t) = \text{echo}\, s \oplus \text{echo}\, t \quad \text{for every } s, t \in \mathbb{S}$$

Solution Part (a).

$$\begin{aligned} \text{echo}() &= () \\ \text{echo}(s, a) &= \text{echo} \oplus (a, a) \quad \text{for each } s \in \mathbb{S} \text{ and } a \in \mathbb{A} \end{aligned}$$

Part (b). Let $s \in \mathbb{S}$, and we will prove this by induction on $t$.

*Base case.* Using the definition of $\oplus$ and echo:

$$\begin{aligned} \text{echo}(s \oplus ()) &= \text{echo}\, s \\ &= \text{echo}\, s \oplus () \\ &= \text{echo}\, s \oplus \text{echo}() \quad \text{using the definition of echo} \end{aligned}$$

*Inductive step.* Let $t \in \mathbb{S}$, and suppose that $\text{echo}(s \oplus t) = \text{echo}\, s \oplus \text{echo}\, t$. Let $a \in \mathbb{A}$. We will show that the longer string $(t, a)$ also has the desired property. Then:

$$\begin{aligned} \text{echo}(s \oplus (t, a)) &= \text{echo}(s \oplus t, a) \\ &= \text{echo}(s \oplus t) \oplus (a, a) && \text{by definition} \\ &= \text{echo}\, s \oplus \text{echo}\, t \oplus (a, a) && \text{by the inductive hypothesis} \\ &= \text{echo}\, s \oplus \text{echo}(t, a) && \text{by definition} \end{aligned}$$

Problem 3

Let $L$ be a signature with one one-place function symbol `s` and one two-place function symbol `f`. Let $S$ be the $L$-structure such that

$$
\begin{aligned}
D_S &= \mathbb{N} \\
[\texttt{s}]_S(n) &= 1 + n && \text{for each } n \in \mathbb{N} \\
[\texttt{f}]_S(m, n) &= \min(m, n) && \text{for each } m, n \in \mathbb{N}
\end{aligned}
$$

So, for example,

$$[\![\texttt{s(f(s(x), x))}]\!]_S(3) = 1 + \min(1 + 3, 3) = 4$$

Use induction to prove:

$$[\![t(x)]\!]_S(n) \geq n \quad \text{for each } L(x)\text{-term } t(x) \text{ and number } n \in \mathbb{N}$$

In other words, for any number $n$, any term of one variable denotes, with respect to $n$, some number which is least $n$.

Solution Let $n$ be any number. We will prove this by induction on terms.

1. For the variable $x$, we have $[\![x]\!]_S(n) = n$, and so in particular $[\![x]\!]_S(n) \geq n$.
2. We have one one-place function symbol `s`. Let $t(x)$ be a term and assume for the inductive hypothesis that $[\![t(x)]\!]_S(n) \geq n$. Then

$$
\begin{aligned}
[\![\texttt{s(}t(x)\texttt{)}]\!]_S(n) &= [s]_S\left([\![t(x)]\!]_S(n)\right) \\
&= 1 + [\![t(x)]\!]_S(n) \\
&> [\![t(x)]\!]_S(n) \\
&\geq n
\end{aligned}
$$

3. We have one two-place function symbol `f`. Let $t_1(x)$ and $t_2(x)$ be terms and assume for the inductive hypothesis that

$$[\![t_1(x)]\!]_S(n) \geq n \quad \text{and} \quad [\![t_2(x)]\!]_S(n) \geq n$$

Then

$$
\begin{aligned}
[\![\texttt{f(}t_1(x)\texttt{,}t_2(x)\texttt{)}]\!]_S(n) &= [f]_S([\![t_1(x)]\!]_S(n), [\![t_2(x)]\!]_S(n)) \\
&= \min\left([\![t_1(x)]\!]_S(n), [\![t_2(x)]\!]_S(n)\right)
\end{aligned}
$$

Since both $[\![t_1(x)]\!]_S(n) \geq n$ and $[\![t_2(x)]\!]_S(n) \geq n$, this holds in particular for whichever of them is smallest. So

$$\min\left([\![t_1(x)]\!]_S(n), [\![t_2(x)]\!]_S(n)\right) \geq n$$

and we are done.

Problem 4

For each of the following sets, either show that it is countable, or else show that it is uncountable. (You may use any facts we have already proved about countable and uncountable sets—so your explanation may be quite short.)

(a) The set $\mathbb{N} \times P\mathbb{N}$, which contains all ordered pairs of a number and a set of numbers.
(b) The set of all (finite) strings of ones and zeros (such as `001110` or `10`).
(c) The set of all functions from $\mathbb{N}$ to $\{0, 1\}$ with *bounded support*, where a function $f : \mathbb{N} \to \{0, 1\}$ has bounded support iff there is some number $n \in \mathbb{N}$ such that $f(k) = 0$ for every number $k \geq n$.

Solution

(a) *Uncountable.* We know that $P\mathbb{N}$ is uncountable, and clearly $P\mathbb{N} \leq \mathbb{N} \times P\mathbb{N}$. Consider, for example, the one-to-one function $f : P\mathbb{N} \to \mathbb{N} \times P\mathbb{N}$ such that $f(X) = (0, X)$ for each $X \in P\mathbb{N}$.

(b) *Countable.* We already know that the set $\mathbb{S}$ of all finite strings is countable, and this is a subset of of $\mathbb{S}$.

(c) *Countable.* One way to show this is to use part (b): we can define a one-to-one function from the set of functions with finite support to the set of all finite strings of zeros and ones, just by deleting all the zeros at the end to get a finite sequence (and then mapping the number 0 to `0` and the number one to `1`).

Another way to show this is to notice that for each $n$, there are only finitely many functions from $\mathbb{N}$ to $\{0, 1\}$ that go to zero for every $k \geq n$. So we can list out all of these functions in an infinite sequence: first the one that is zero everywhere, then we list the ones that are zero for $k \geq 1$, then for $k \geq 2$, and so on.

Problem 5

Let $L$ be a signature containing one name `c` and one two-place predicate `R`. For each of the following sets of $L$-sentences, either show that it is consistent by providing a model, or else prove that it is inconsistent.

(a) {`∀x R(c,x)`, `¬∀x R(x,c)`}
(b) {`∀x R(c,x)`, `∀x ¬R(x,c)`}

Solution

(a) This set is consistent. Here is a model:

$$D_S = \{0, 1\}$$
$$[\texttt{c}]_S = 0$$
$$[\texttt{R}]_S = \{(0,0), (0,1)\}$$

(There are many other models that would work just as well.)

We see that for every $d \in D_S$, the pair $([\texttt{c}]_S, d) = (0, d) \in [\texttt{R}]_S$. Using our semantic definitions, this implies that `∀x R(c,x)` is true in $S$. Also, the pair $(1, 0)$ is not in $[\texttt{R}]_S$; so it is not the case that for every $d \in D_S$ we have $(d, [\texttt{c}]_S) \in [\texttt{R}]_S$. So `¬∀x R(x,c)` is also true in $S$.

(b) This set is inconsistent. Suppose for contradiction that $S$ is a model of this set. So `∀x R(c,x)` is true in $S$, and thus the semantics for first-order logic imply that for every $d \in D_S$,

$$([\texttt{c}]_S, d) \in [\texttt{R}]_S$$

In particular,

$$([\texttt{c}]_S, [\texttt{c}]_S) \in [\texttt{R}]_S$$

Also, `∀x ¬R(x,c)` is true in $S$, which implies that for every $d \in D_S$,

$$(d, [\texttt{c}]_S) \notin [\texttt{R}]_S$$

So, in particular,

$$([\texttt{c}]_S, [\texttt{c}]_S) \notin [\texttt{R}]_S$$

This is a contradiction, so there is no such structure $S$.

Problem 6

Use the definitions to prove each of the following:

(a) Let $X$ and $Y$ be sets of formulas and let $A$ be a formula. If $X \cup Y$ is consistent and $X \vDash A$, then $Y \cup \{A\}$ is consistent.

(b) Let $X$ be a set of formulas, and let $A, B, C$ be formulas.

$$\text{If} \quad X, \neg A \vDash C \quad \text{and} \quad X, \neg B \vDash C \quad \text{then} \quad X, \neg(A \mathbin{\&} B) \vDash C$$

Solution

(a) Let $X$ and $Y$ be sets of formulas and let $A$ be a formula, and suppose that $X \cup Y$ is consistent and $X \vDash A$. The first assumption means that $X \cup Y$ has a model, say $S$. Since every sentence in $X \cup Y$ is true in $S$, in particular every sentence in $X$ is true in $S$, which means $S$ is a model of $X$. Thus, since $X \vDash A$, we must have $A$ true in $S$. Furthermore, we know that every sentence in $Y$ is true in $S$. So $S$ is a model of $Y \cup \{A\}$, which shows that $Y \cup \{A\}$ is consistent.

(b) Let $X$ be a set of formulas, let $A, B, C$ be formulas, and suppose

$$X, \neg A \vDash C \quad \text{and} \quad X, \neg B \vDash C$$

Let $S$ be any model of $X \cup \{\neg(A \ \& \ B)\}$. Since $\neg(A \ \& \ B)$ is true in $S$, it follows that either (i) $A$ is not true in $S$, or (ii) $B$ is not true in $S$.

Suppose (i). Then $\neg A$ is true in $S$, and so $S$ is a model of $X \cup \{\neg A\}$. Thus, since $X, \neg A \vDash C$, we must have $C$ true in $S$.

Likewise, supposing (ii), then $S$ is a model of $X \cup \{\neg B\}$. Since $X, \neg B \vDash C$, we have $C$ true in $S$ in this case as well.

Thus $C$ is true in every model of $X \cup \{\neg(A \ \& \ B)\}$, and thus $X, \neg(A \ \& \ B) \vDash C$.