Midterm Solutions

The Limits of Logic

Fall 2022

Problem 1

Let A and B be any sets, and let $f : A \to B$ be any function. Let $E : B \to PA$ be the function defined as follows:

$$E(b) = \{a \in A \mid f(a) = b\}$$

Prove that for every $a \in A$, there is exactly one $b \in B$ such that $a \in E(b)$.

Solution First we will show that there is *at least one* such $b \in B$, and then we will show that there is *at most one* such $b \in B$.

Existence: We have in particular

$$E(f(a)) = \{a' \in A \mid f(a') = f(a)\}$$

Clearly $a \in E(f(a))$ (since $f(a) = f(a)$). So there is some $b \in B$ such that $a \in E(b)$.

Uniqueness: Suppose that there are $b, b' \in B$ such that $a \in E(b)$ and $E(b')$. This means that $f(a) = b$ and $f(a) = b'$. This implies that $b = b'$.

Problem 2

Let the set of *even palindromes* be recursively defined as follows:

- The empty string () is an even palindrome.
- For any even palindrome s, and any symbol $a \in \mathbb{A}$ in the standard alphabet, the string $(a) \oplus s \oplus (a)$ is an even palindrome.

(To be clear, (a) is the length-one string that consists of just the symbol a.)

Part (a) Prove by induction on the definition of *even palindrome* that, for every even palindrome s, the length of s is an even number.

Solution Base case: the length of () is zero, which is an even number.

Inductive step: Let s be any even palindrome, and suppose the length of s is even. We will show that the length of $(a) \oplus s \oplus (a)$ is also even. In fact,

$$\text{length}((a) \oplus s \oplus (a)) = 1 + \text{length}\, s + 1 = \text{length}\, s + 2$$

Since by the inductive hypothesis the length of s is even, this is also even.

Part (b) Write down a recursive definition of the function *reverse*, which takes any string s to the string that has the same symbols as s in reverse order. For example:

$$\begin{aligned}
\text{reverse}\, \texttt{ABC} &= \texttt{CBA} \\
\text{reverse}\, \texttt{ABBCA} &= \texttt{ACBBA} \\
\text{reverse}\, \texttt{B} &= \texttt{B} \\
\text{reverse}() &= ()
\end{aligned}$$

Solution

$$\begin{aligned}
\text{reverse}() &= () \\
\text{reverse}(a : s) &= (\text{reverse}\, s) \oplus (a)
\end{aligned}$$

Part (c) Use your definition to prove by induction that for any string s, $s \oplus \text{reverse}\, s$ is an even palindrome.

Solution For the base case, we show that $() \oplus \text{reverse}()$ is an even palindrome. This is true, since $() \oplus \text{reverse}() = () \oplus () = ()$.

For the inductive step, suppose that $s \oplus \text{reverse}\, s$ is an even palindrome. We will show that $(a : s) \oplus \text{reverse}(a : s)$ is also an even palindrome. In fact, using our definition

$$\begin{aligned}
(a : s) \oplus \text{reverse}(a : s) &= (a : s) \oplus \text{reverse}\, s \oplus (a) \\
&= (a) \oplus (s \oplus \text{reverse}\, s) \oplus (a)
\end{aligned}$$

Since $s \oplus \text{reverse}\, s$ is an even palindrome by the inductive hypothesis, the definition of even palindromes tells us that this is also an even palindrome.

Problem 3

For each of the following sets, say whether it is **countable**, or **uncountable.** Explain each answer in sentence or two.

(a) The set of all finite sequences of finite sequences of numbers.

(b) The set of all finite sets of strings.

(c) The set of all infinite sets of strings.

(d) The set of all sets of even palindromes (as defined in problem 2).

Solution

(a) **Countable.** We know that for any countable set A, the set A^* of all finite sequences of elements of A is also countable. So, since $\mathbb{N}$ is countable, $\mathbb{N}^*$ is countable, and then applying the same fact again, $\mathbb{N}^{**}$ is also countable.

(b) **Countable.** We know that the set of all finite *sequences* of strings is countable (by the same fact we used in part (a)). Furthermore, there are no more finite *sets* of strings than there are finite *sequences* of strings. We can see this because there is an onto function that takes each finite sequence of strings to a finite set—namely, the set of all of the elements of that sequence. (For example, this takes $(\mathtt{A}, \mathtt{B}, \mathtt{A})$ to $\{\mathtt{A}, \mathtt{B}\}$.)

(c) **Uncountable.** The set $P\mathbb{S}$ of *all* sets of strings is uncountable, and by part (b) the set of *finite* sets of strings is countable—call this $P_0\mathbb{S}$. Since $P\mathbb{S}$ is the union of $P_0\mathbb{S}$ and the set of infinite sets of strings, the latter set must *not* be countable.

(d) **Uncountable.** There are infinitely many even palindromes: for example each of the strings $\mathtt{AA}$, $\mathtt{AAAA}$, $\mathtt{AAAAAA}$, … is an even palindrome. We know that the set of all subsets of an infinite set is uncountable.

Problem 4 Let L be a signature containing one constant symbol $\mathtt{c}$, one one-place function symbol $\mathtt{f}$, and no other basic symbols. Let S and S' be L-structures. Let $h : D_S \to D_{S'}$ be any function from the domain of S to the domain of S'.

Suppose that

$$\mathtt{c}_{S'} = h(\mathtt{c}_S)$$

That is, the denotation of $\mathtt{c}$ in S' is the result of applying the function h to the denotation of $\mathtt{c}$ in S.

Suppose also that for every $d \in D_S$,

$$\mathtt{f}_{S'}\big(h(d)\big) = h\big(\mathtt{f}_S(d)\big)$$

In other words, for any pair $(d, d') \in D_S \times D_S$,

$$\text{If} \quad \mathtt{f}_S(d) = d' \quad \text{then} \quad \mathtt{f}_{S'}\big(h(d)\big) = h(d')$$

In this sense, the function h takes the extension of $\mathtt{f}$ in S to the extension of $\mathtt{f}$ in S'.

Part (a) Prove by induction that, for any (closed) L-term t, if t denotes d in S, then t denotes $h(d)$ in S'. In other words, for any L-term t,

$$[\![t]\!]_{S'} = h\big([\![t]\!]_S\big)$$

Solution *Base case.*

$$h([\![\mathtt{c}]\!]_S) = h(\mathtt{c}_S) = \mathtt{c}_{S'} = [\![\mathtt{c}]\!]_{S'}$$

Inductive step. Let t be any term. Suppose that $[\![t]\!]_{S'} = h\big([\![t]\!]_S\big)$. We will show that $h([\![\mathtt{f}(t)]\!]_S) = [\![\mathtt{f}(t)]\!]_{S'}$. In fact:

$$\begin{aligned} h([\![\mathtt{f}(t)]\!]_S) &= h(\mathtt{f}_S[\![t]\!]_S) \\ &= \mathtt{f}_{S'}(h([\![t]\!]_S)) \\ &= \mathtt{f}_{S'}([\![t]\!]_{S'}) \\ &= [\![\mathtt{f}(t)]\!]_{S'} \end{aligned}$$

Part (b) Use part (a) to prove that, for any L-terms a and b, if $[\![a]\!]_S = [\![b]\!]_S$, then $[\![a]\!]_{S'} = [\![b]\!]_{S'}$.

Solution If $[\![a]\!]_S = [\![b]\!]_S$, then

$$h([\![a]\!]_S) = h([\![b]\!]_S)$$

By part (a), the left-hand side is $[\![a]\!]_{S'}$ and the right-hand side is $[\![b]\!]_{S'}$. So these are equal.

Problem 5

Consider a signature L that contains one two-place predicate symbol F, and no other basic symbols. For each of the following sets of L-sentences, either prove that it is consistent, or prove that it is inconsistent.

Part (a) The set containing the following two sentences:

```
∃x ∀y F(x,y)
∀x ¬∃y F(x,y)
```

Solution **Inconsistent.** Let S be any structure, and suppose first that `∀x ¬∃y F(x,y)` is true in S. That means that for every $d_1 \in D_S$, it is not the case that there is some $d_2 \in D_S$ such that (d_1, d_2) is in F_S (the extension of F in S). In other words, there are no d_1 and d_2 in D_S such that $(d_1, d_2) \in F_S$; the extension of F is empty.

Suppose next that `∃x ∀y F(x,y)` is true in S. That means that there is some $d_1 \in D_S$ such that for every $d_2 \in D_S$, (d_1, d_2) is in F_S (the extension of F in S). In particular, $(d_1, d_1) \in F_S$. So the extension of F_S is not empty.

This is a contradiction, so both suppositions cannot be true: that is, there is no structure in which `∀x ¬∃y F(x,y)` and `∃x ∀y F(x,y)` are both true. So the set containing both sentences is inconsistent.

Part (b) The set containing the following two sentences:

```
¬∃x ∀y F(x,y)
∀x ∃y F(x,y)
```

Solution **Consistent.** We can show this by providing a model. We can let S be a structure such that

$$D_S = \{1, 2\}$$
$$F_S = \{(1, 1), (2, 2)\}$$

In a picture:

```
1       2
•       •
↻       ↻
```

- `¬∃x ∀y F(x,y)` is true in S: there is no $d_1 \in D_S$ such that, for every $d_2 \in D_S$, $(d_1, d_2) \in F_S$. In the case of 1, we have $(1,2) \notin F_S$, and in the case of 2, we have $(2,1) \notin F_S$.
- `∀x ∃y F(x,y)` is true in S: for each $d_1 \in D_S$, there is some $d_2 \in D_S$ such that $(d_1, d_2) \in F_S$. In the case of 1 we have $(1,1) \in F_S$ and in the case of 2 we have $(2,2) \in F_S$.

Part (c) The set containing the following two sentences:

```
∃x ∀y F(x,y)
¬∀x ∃y F(x,y)
```

Solution **Consistent.** Again we provide a model. Let S be a structure such that

$$D_S = \{1, 2\}$$
$$F_S = \{(1,1), (1,2)\}$$

In a picture:

```
1       2
•  →    •
↻
```

- `∃x ∀y F(x,y)` is true in S: there is some $d \in D_S$, namely 1, such that for every $d_2 \in D_S$ we have $(d_1, d_2) \in F_S$. That is, $(1,1)$ and $(1,2)$ are both in F_S.
- `¬∀x ∃y F(x,y)` is true in S: not every $d_1 \in D_S$ is such that for some $d_2 \in D_S$ we have $(d_1, d_2) \in F_S$. In particular, 2 is a counterexample, since neither $(2,1)$ nor $(2,2)$ is in F_S.

Problem 6

Prove the following:

Part (a) For any set of sentences X and any sentences A, and B, if $X \cup \{A, B\}$ is inconsistent, then $X \vDash \neg(A \wedge B)$.

Solution Suppose that $X \cup \{A, B\}$ is inconsistent, and let S be any model of X. Then, since $X \cup \{A, B\}$ has no models, either A is false in S, or B is false in S. In either case, $A \wedge B$ is false in S, and thus $\neg(A \wedge B)$ is true in S. This shows that $\neg(A \wedge B)$ is true in every model of X, which means $X \vDash \neg(A \wedge B)$.

Part (b) For any set of sentences X, any terms a and b, and any formula $A(x)$, if $X \cup \{a = b, A(a)\}$ is consistent, then $X \cup \{A(b)\}$ is consistent.

Solution Suppose $X \cup \{a = b, A(a)\}$ is consistent: that is, it has a model. So let S be such a model. Then, since $a = b$ is true in S, we have $[\![a]\!]_S = [\![b]\!]_S$, and since $A(a)$ is true in S, $A(x)$ is true of $[\![a]\!]_S$ in S (by the Substitution Lemma). Then $A(x)$ is also true of $[\![b]\!]_S$ in S. So $A(b)$ is true in S (by the Substitution Lemma again). Thus S is a model of $X \cup \{A(b)\}$. Since this set has a model, it is consistent.