Midterm

The Limits of Logic

Due 2pm Oct 18, 2023

Turn in answers to any five out of the following six problems. Each problem is worth 8 points, for 40 total points possible.

Explain your reasoning clearly for each question. You can use any of the facts we've already proved in class without further justification, but please be explicit about which facts you are using.

Problem 1

Let A be a set, and let X be a set of subsets of A. (That is, $X \subseteq PA$.) We define the relation $\sqsubseteq$ as follows:

> For any $a, b \in A$, let $a \sqsubseteq b$ iff, for every set $U \in X$, if $a \in U$, then $b \in U$.

(a) Prove that for any $a, b, c \in A$, if $a \sqsubseteq b$ and $b \sqsubseteq c$ then $a \sqsubseteq c$.

(b) Suppose that for any $a, b \in A$, there is some $c \in A$ such that we have both $a \sqsubseteq c$ and $b \sqsubseteq c$. (This is called the *directed set property.*) Prove that any two non-empty sets $U, V \in X$ have some element in common: in other words, the intersection $U \cap V \neq \varnothing$.

Problem 2

(a) Write down a recursive definition of a function echo : $\mathbb{S} \to \mathbb{S}$ that takes each string to a corresponding string with each symbol repeated. For example:

$$\text{echo}\,\texttt{ABC} = \texttt{AABBCC}$$
$$\text{echo}\,\texttt{AAB} = \texttt{AAAABB}$$
$$\text{echo}\,\texttt{BAAAC} = \texttt{BBAAAAAACC}$$
$$\text{echo}() = ()$$

(b) Use your definition to prove by the following by induction:

$$\text{echo}(s \oplus t) = \text{echo}\, s \oplus \text{echo}\, t \quad \text{for every } s, t \in \mathbb{S}$$

Problem 3

Let L be a signature with one one-place function symbol `s` and one two-place function symbol `f`. Let S be the L-structure such that

$$\begin{aligned} D_S &= \mathbb{N} \\ [\texttt{s}]_S(n) &= 1 + n && \text{for each } n \in \mathbb{N} \\ [\texttt{f}]_S(m, n) &= \min(m, n) && \text{for each } m, n \in \mathbb{N} \end{aligned}$$

So, for example,

$$[\![\texttt{s(f(s(x), x))}]\!]_S(3) = 1 + \min(1 + 3, 3) = 4$$

Use induction to prove:

$$[\![t(x)]\!]_S(n) \geq n \quad \text{for each } L(x)\text{-term } t(x) \text{ and number } n \in \mathbb{N}$$

In other words, for any number n, any term of one variable in this language denotes, with respect to n, some number which is least n.

Problem 4

For each of the following sets, either show that it is countable, or else show that it is uncountable. (You may use any facts we have already proved about countable and uncountable sets—so your explanation may be quite short.)

(a) The set $\mathbb{N} \times P\mathbb{N}$, which contains all ordered pairs of a number and a set of numbers.

(b) The set of all (finite) strings of ones and zeros (such as `001110` or `10`).

(c) The set of all functions from $\mathbb{N}$ to $\{0, 1\}$ with *bounded support*, where a function $f : \mathbb{N} \to \{0, 1\}$ has bounded support iff there is some number $n \in \mathbb{N}$ such that $f(k) = 0$ for every number $k \geq n$.

Problem 5

Let L be a signature containing one name `c` and one two-place predicate `R`. For each of the following sets of L-sentences, either show that it is consistent by providing a model, or else prove that it is inconsistent.

(a) {`∀x R(c,x)`, `¬∀x R(x,c)`}

(b) {`∀x R(c,x)`, `∀x ¬R(x,c)`}

Problem 6

Use the definitions to prove each of the following:

(a) Let X and Y be sets of formulas and let A be a formula. If $X \cup Y$ is consistent and $X \vDash A$, then $Y \cup \{A\}$ is consistent.

(b) Let X be a set of formulas, and let A, B, C be formulas.

$$\text{If} \quad X, \neg A \vDash C \quad \text{and} \quad X, \neg B \vDash C \quad \text{then} \quad X, \neg(A \ \& \ B) \vDash C$$