Final exam

Essay question

The Limits of Logic

Due 4:30pm Dec 8, 2023

Your answer to this question should be **no more than 2 pages** (typed). Shorter is fine: just answer the question as clearly and straightforwardly as you can. **Choose one** of the following prompts.

Option 1

Suppose someone argues as follows:

> There should be no gap between *proof* and *truth* in mathematics. Of course, I grant that, because of Gödel's theorems, there are certain mathematical statements which are not provable from our current axioms, such as ZFC. All this shows is that we need to add *new* axioms to our current theory, in order to come up with a complete theory. For example, we might add the further axiom that ZFC is consistent. Once we have expanded our set of axioms in a reasonable way, in principle we will be able to prove every true mathematical statement.

What is problematic about this idea?

Option 2

Hilbert (1925, "On the Infinite") argued that if we are going to rely on reasoning about infinite sets (such as the set of all numbers), we must justify this in terms of "finitistic" reasoning, which does not quantify unrestrictedly over any infinite domain. He proposed that we could do this by proving that reasoning about infinite sets never leads to false conclusions in "finitistic" mathematics (for example, that $2 + 2 = 5$)—and furthermore, that we could prove this using only "finitistic" reasoning.

In other words, what Hilbert wanted was to prove that a sufficiently strong theory of arithmetic (such as PA) is consistent, where all of the premises of this consistency proof are true Δ_0 sentences (sentences which use only bounded quantifiers).

Explain why Hilbert's proposal is impossible to carry out, appealing to Gödel's Second Incompleteness Theorem.

Option 3

Here is another way of formulating Gödel's First Incompleteness Theorem:

> Let T be a sufficiently strong, effectively axiomatizable theory, which consists of only sentences that are true in the standard model of arithmetic $\mathbb{N}$. Then there is a *Gödel sentence* G_T such that G_T is true (in $\mathbb{N}$), and G_T is not a theorem of T.

Make sure you understand why this is true. Then consider the following argument (roughly based on Lucas 1961, "Minds, Machines, and Gödel"):

> Let K be the the set of all sentences (in the language of arithmetic) that human reasoners can *know* to be true. It is clear that K is sufficiently strong—since we can know that each theorem of the minimal theory of arithmetic Q is true. Also, K consists only of truths, since no one can know anything false. Now, suppose that K was also effectively axiomatizable. Then there would have to be a Gödel sentence G_K which is true, but which we could not know to be true. But we human reasoners *can* know that Gödel's First Incompleteness Theorem is true, and so we would be able to know that G_K is true! This is a contradiction. So K must *not* be effectively axiomatizable, after all. In this sense, human reasoning is not "mechanistic."

Is this argument convincing? Why or why not?